THE FOURTH INTERNATIONAL MATHEMATICAL OLYMPIAD
Blagoveshchensk – Russia, 16 March 2024

Problem 1 ***(9 points)***

Prove the inequality

$$\left(1+\frac{1}{4}\right)\left(1+\frac{1}{8}\right)\dots\left(1+\frac{1}{2^n}\right)<2,\quad n\geq 2.$$

Problem 2 ***(11 points)***

Calculate the indefinite integral

$$I=\int\frac{\cos x+\sin x}{5\cos^2 x-2\sin 2x+2\sin^2 x}dx.$$

Problem 3 ***(9 points)***

Calculate the definite integral

$$\int_0^1\frac{x^{2023}-1}{\ln x}dx.$$

Problem 4 ***(10 points)***

Find a sum of the number series

$$\sum_{n=1}^{\infty}\frac{n^2}{3^n}.$$

Problem 5 ***(9 points)***

Find all real solutions to the equation

$$(2x^3+x-3)^3=3-x^3.$$

Problem 6 ***(9 points)***

Find a conditional extremum of the function

$$u = xyz$$

with the following coupling equations

$$xy + yz + zx = 9, \;\; x + y + z = 6.$$

Problem 7 ***(10 points)***

Find a general solution to the differential equation

$$y^2 dx + (e^x - y)dy = 0.$$

Problem 8 ***(11 points)***

Find a general solution to the system of nonlinear differential equations

$$\begin{cases} y' = \dfrac{z}{x}, \\ z' = \dfrac{z(y + 2z - 1)}{x(y - 1)}. \end{cases}$$

Problem 9 ***(11 points)***

Calculate *n*-th order determinant

$$\begin{vmatrix} 3 & 8 & 0 & 0 & \cdots & 0 & 0 \\ 2 & 6 & 9 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 6 & 9 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 6 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 6 & 9 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 6 \end{vmatrix}.$$

Problem 10 ***(11 points)***

Random variables ξ_1, ξ_2, ξ_3 are independent and uniformly distributed on the interval $[0;\ 1]$.

Find the probability $P(\xi_1 + \xi_2 + \xi_3 \le x)$ for $x \ge 0$.