ТРЕТЬЯ МЕЖДУНАРОДНАЯ МАТЕМАТИЧЕСКАЯ ОЛИМПИАДА
БЛАГОВЕЩЕНСК – РОССИЯ, 25 марта 2023 г.

THE THIRD INTERNATIONAL MATHEMATICAL OLYMPIAD
Blagoveshchensk – Russia, 25 March 2023

第三届国际数学奥林匹克竞赛

布拉戈维申斯克-俄罗斯，2023年3月25日

Problem statements and solutions

Problem 1 ***(8 points)***
Find the solution of the system of equations

$$\begin{cases} x + \dfrac{3x - y}{x^2 + y^2} = 3, \\ y - \dfrac{x + 3y}{x^2 + y^2} = 0. \end{cases}$$

Answer: $x_1 = 2, \quad y_1 = 1; \; x_2 = 1, \; y_2 = -1.$

Solution:
Multiply the first equation by y, the second – by x, and add:

$$(1) \cdot y + (2) \cdot x \quad \Rightarrow \quad 2xy - 1 = 3y \quad \Rightarrow \quad y = \frac{1}{2x - 3}.$$

Next, multiply the first equation by x, the second – by y, and subtract:

$$(1) \cdot x - (2) \cdot y \quad \Rightarrow \quad x^2 - y^2 + 3 = 3x.$$

Then we get:

$$x^2 - 3x + 3 = \frac{1}{(2x - 3)^2} \quad \Rightarrow \quad \left(x - \frac{3}{2}\right)^2 + \frac{3}{4} = \frac{1}{4\left(x - \frac{3}{2}\right)^2};$$

$$t = \left(x - \frac{3}{2}\right)^2 \quad \Rightarrow \quad 4t^2 + 3t - 1 = 0, \;\; D = 9 + 16 = 25;$$

$$t_{1,2} = \frac{-3 \pm 5}{8} \Rightarrow t = \frac{1}{4}.$$

Finally we find a solution:

$$x - \frac{3}{2} = \pm\frac{1}{2} \quad \Rightarrow \quad x_1 = 2,\ x_2 = 1,\ y_1 = 1,\ y_2 = -1.$$

Problem 2 ***(10 points)***

A tangent line is drawn to the graph of the function $y = -(x^2/12) + x - 16/3$. This line intersects the graph of the function $y = 3|x+6| - 7/3$ at points A and B. Find the radius of a circle circumscribed around a triangle with vertices at points A, B and $C(-6;\ -7/3)$, if $\angle CAB = 2arccos\left(3/\sqrt{10}\right) + \angle CBA$.

Answer: $R = 13\sqrt{10}/16$

Solution:

We consider ΔABC and note $\angle ACB = 2arccos\left(3/\sqrt{10}\right) \Rightarrow \angle CAB = 90^0$.

Compose the equation of the tangent to the parabola:

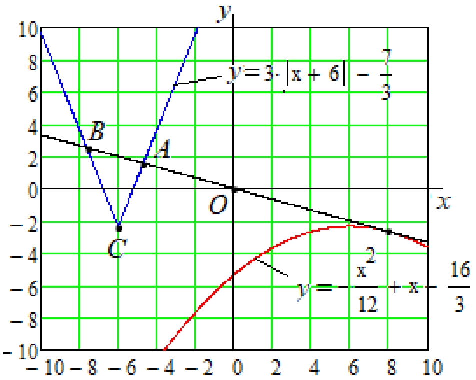
$$y(x) = y(x_0) + y'(x_0)\cdot(x - x_0),\ \ y'(x_0) = -\frac{x_0}{6} + 1 = -\frac{1}{3} \ \Rightarrow\ x_0 = 8,\ y(x) = -\frac{1}{3}x.$$

Determine the coordinates of the point B:

$$y_B = -3(x_B + 6) - \frac{7}{3},\ \ y_B = -\frac{1}{3}x_B \ \Rightarrow\ x_B = -\frac{61}{8},\ y_B = \frac{61}{24}.$$

Next, the diameter of the circumscribed circle coincides with the hypotenuse BC:

$$2R = BC = \sqrt{(x_C - x_B)^2 + (y_C - y_B)^2} = \sqrt{\left(-6 + \frac{61}{8}\right)^2 + \left(-\frac{7}{3} - \frac{61}{24}\right)^2} = \frac{13\sqrt{10}}{8}.$$



Problem 3 ***(9 points)***
The matrices $A_{3\times 2}$ and $B_{2\times 3}$ are such that

$$AB=\begin{pmatrix} 8 & 2 & -2 \\ 2 & 5 & 4 \\ -2 & 4 & 5 \end{pmatrix}.$$

Find the matrix BA.

Answer: $BA=9E$, where E is a unit matrix with dimensions 2×2.

Solution:
Let $A'_{2\times 3}$ is a matrix such that $A'A=E$ and $B'_{3\times 2}$ a matrix such that $BB'=E$, where E is a unit matrix of size 2×2. It is not difficult to show that such matrices A' and B' exist, and not the only ones.

Next, we note that $(AB)\cdot(AB)=9(AB)$. Multiplying both parts of this equality on the left by A', and on the right by B', we get

$$A'\cdot(AB)\cdot(AB)\cdot B'=9A'\cdot(AB)\cdot B'\Longrightarrow$$
$$\Longrightarrow(A'A)\cdot(BA)\cdot(BB')=9(A'A)\cdot(BB')\Longrightarrow\ BA=9E.$$

Problem 4 ***(11 points)***
Find the general solution or the general integral of the equation

$$\frac{dy}{dx}=\frac{xy}{y^3+x^2y+x^2}.$$

Answer: $y^2e^{2y}=C(x^2+y^2)$.

Solution:
a)

$$(y^3+x^2y+x^2)dy=xydx\quad\Longrightarrow\quad x(ydx-xdy)=y(x^2+y^2)dy\quad\Longrightarrow$$

$$\Longrightarrow\ \frac{ydx-xdy}{x^2+y^2}=\frac{y}{x}dy\quad\Longrightarrow\quad -d\left(\operatorname{arctg}\frac{y}{x}\right)=\frac{y}{x}dy\quad\Longrightarrow$$

$$\Longrightarrow\quad -d(\operatorname{arctg}z)=zdy,\quad z=\frac{y}{x}\quad\Longrightarrow\quad -\frac{d(\operatorname{arctg}z)}{z}=dy\quad\Longrightarrow$$

$$\Longrightarrow\quad -\frac{dz}{z(1+z^2)}=dy\ \Longrightarrow y=-\int\frac{dz}{z}+\int\frac{zdz}{1+z^2}+\tilde{C}\ \Longrightarrow$$

$$\Longrightarrow y=-\ln|z|+\frac{1}{2}\ln|1+z^2|+\tilde{C}\quad\Longrightarrow\quad z^2=(1+z^2)Ce^{-2y}\quad\Longrightarrow$$

$$\Longrightarrow \quad y^2e^{2y} = C(x^2+y^2).$$

b)

$$(y^3+x^2y+x^2)dy = xydx \quad \Longrightarrow \quad x(ydx-xdy) = y(x^2+y^2)dy \quad \Longrightarrow$$

$$\Longrightarrow \quad \frac{ydx-xdy}{x^2+y^2} = \frac{y}{x}dy \quad \Longrightarrow$$

$$\Longrightarrow \frac{-d\left(\frac{y}{x}\right)}{1+\left(\frac{y}{x}\right)^2} = \frac{y}{x}dy, \;\; z = \frac{y}{x} \quad \Longrightarrow \quad -\frac{dz}{z(1+z^2)} = dy \quad \Longrightarrow$$

$$\Longrightarrow \quad y = -\int\frac{dz}{z} + \int\frac{zdz}{1+z^2} + C \;\Longrightarrow$$

$$\Longrightarrow \quad y = -\ln|z| + \frac{1}{2}\ln|1+z^2| + C \quad \Longrightarrow \quad z^2 = (1+z^2)Ce^{-2y} \quad \Longrightarrow$$

$$\Longrightarrow \quad y^2e^{2y} = C(x^2+y^2).$$

Problem 5 ***(10 points)***

Find the limit of the sum

$$\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n}\right).$$

Answer: ln2.

Solution:

Denote

$$\Delta x_k = \Delta x = \frac{1}{n}, \;\; x_k = k\Delta x = \frac{k}{n},$$

then

$$\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n}\right) =$$

$$= \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\cdot\frac{1}{n}+\frac{1}{1+\frac{2}{n}}\cdot\frac{1}{n}+\cdots+\frac{1}{1+\frac{n}{n}}\cdot\frac{1}{n}\right) =$$

$$= \lim_{n\to\infty}\left(\frac{1}{1+x_1}\cdot\Delta x+\frac{1}{1+x_2}\cdot\Delta x+\cdots+\frac{1}{1+x_n}\cdot\Delta x\right) = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{1+x_k}\cdot\Delta x_k =$$

$$= \int_0^1\frac{dx}{1+x} = ln(1+x)|_0^1 = \ln 2.$$

Problem 6 ***(11 points)***

The sequence $\{u_n\}$ is given recurrently:

$$u_0 = 1,\ u_1 = 2,\ u_2 = 4, \qquad u_{n+1} = u_{n-2} + 2u_{n-1} + u_n, \qquad n \geq 2.$$

Find the sum of the series

$$\sum_{n=0}^{+\infty} \frac{u_n}{10^n}.$$

Answer: 1100/879.

Solution:

Let's denote S is the sum of the series. We find

$$S = \sum_{n=0}^{+\infty} \frac{u_n}{10^n} = u_0 + \frac{u_1}{10} + \frac{u_2}{10^2} + \sum_{n=3}^{+\infty} \frac{u_n}{10^n} = 1 + \frac{2}{10} + \frac{4}{10^2} + \sum_{n=2}^{+\infty} \frac{u_{n+1}}{10^{n+1}} =$$

$$= \frac{124}{10^2} + \sum_{n=2}^{+\infty} \frac{u_{n-2} + 2u_{n-1} + u_n}{10^{n+1}} =$$

$$= \frac{124}{10^2} + \sum_{n=2}^{+\infty} \frac{u_{n-2}}{10^{n+1}} + 2 \cdot \sum_{n=2}^{+\infty} \frac{u_{n-1}}{10^{n+1}} + \sum_{n=2}^{+\infty} \frac{u_n}{10^{n+1}} =$$

$$= \frac{124}{10^2} + \sum_{n=0}^{+\infty} \frac{u_n}{10^{n+3}} + 2 \cdot \sum_{n=1}^{+\infty} \frac{u_n}{10^{n+2}} + \sum_{n=2}^{+\infty} \frac{u_n}{10^{n+1}} =$$

$$= \frac{124}{10^2} + \frac{1}{10^3} \cdot \sum_{n=0}^{+\infty} \frac{u_n}{10^n} + \frac{2}{10^2} \cdot \sum_{n=1}^{+\infty} \frac{u_n}{10^n} + \frac{1}{10} \cdot \sum_{n=2}^{+\infty} \frac{u_n}{10^n} =$$

$$= \frac{124}{10^2} + \frac{1}{10^3} \cdot \sum_{n=0}^{+\infty} \frac{u_n}{10^n} + \frac{2}{10^2} \cdot \left(\sum_{n=0}^{+\infty} \frac{u_n}{10^n} - u_0 \right) + \frac{1}{10} \cdot \left(\sum_{n=0}^{+\infty} \frac{u_n}{10^n} - u_0 - \frac{u_1}{10} \right) =$$

$$= \frac{124}{10^2} + \frac{1}{10^3} \cdot S + \frac{2}{10^2} \cdot (S - 1) + \frac{1}{10} \cdot \left(S - 1 - \frac{2}{10} \right) =$$

$$= \frac{124}{10^2} + \frac{1}{10^3} \cdot S + \frac{2}{10^2} \cdot S - \frac{2}{10^2} + \frac{1}{10} \cdot S - \frac{1}{10} - \frac{2}{10^2}.$$

Then we get

$$\left(1 - \frac{1}{10^3} - \frac{2}{10^2} - \frac{1}{10}\right) \cdot S = \frac{124}{10^2} - \frac{2}{10^2} - \frac{1}{10} - \frac{2}{10^2}.$$

$$\frac{879}{10^3} \cdot S = \frac{110}{10^2}; \quad S = \frac{110}{10^2} \cdot \frac{10^3}{879} = \frac{1100}{879}.$$

Problem 7 ***(9 points)***

Four players take turns tossing a coin, which each time with the same probability of 0.5 drops out «heads» or «tails».

The winner is the player who gets the «heads» for the first time.

Determine the probability of winning each of the players.

Answer: $8/15, 4/15, 2/15, 1/15.$

Solution:

The first participant has the opportunity to win:

- at the first toss with a probability of 0,5;
- on the fifth toss with probability $(0{,}5)^5$ (the first four tosses must be unsuccessful),
- at the ninth toss with a probability $(0{,}5)^9$, etc.

All these are incompatible events, so the total probability $P(A_1)$ of the first participant winning:

$$P(A_1) = 0{,}5 + (0{,}5)^5 + (0{,}5)^9 + \cdots = 0{,}5 \cdot \frac{1}{1-(0{,}5)^4} = \frac{8}{15}.$$

Similarly, the second participant has the opportunity to win:

- on the second toss with probability $(0{,}5)^2$;
- at the sixth toss with a probability $(0{,}5)^6$, etc.

The total probability $P(A_2)$ of winning the second participant:

$$P(A_2) = (0{,}5)^2 + (0{,}5)^6 + (0{,}5)^{10} + \cdots = (0{,}5)^2 \cdot \frac{1}{1-(0{,}5)^4} = \frac{4}{15}.$$

Similarly,

$$P(A_3) = \frac{2}{15}, \qquad P(A_4) = \frac{1}{15}.$$

Problem 8 ***(13 points)***

Calculate the definite integral

$$I = \int_0^{2\pi} \frac{dx}{(a + b\cos^2 x)^2} \qquad (a > 0, b > 0).$$

Answer: $\frac{\pi(2a+b)}{[a(a+b)]^{3/2}}.$

Solution:

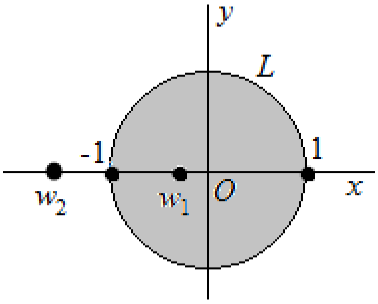
The calculation of the original real integral is reduced to the calculation of the contour integral in the complex plane:

$$z = e^{ix},\ \ dz = ie^{ix}dx = izdx,\ \ \cos x = \frac{e^{ix}+e^{-ix}}{2} = \frac{z^2+1}{2z},\ \ L = \{z: |z| = 1\};$$

$$I = \oint_L \frac{dz}{iz\left(a + b\left(\frac{z^2+1}{2z}\right)^2\right)^2} = -16i\oint_L \frac{z^3dz}{(bz^4 + 2(2a+b)z^2 + b)^2} =$$

$$= -\frac{8i}{b^2}\oint_L \frac{z^2d(z^2)}{\left(z^4 + 2\left(2\frac{a}{b}+1\right)z^2 + 1\right)^2} = \begin{pmatrix} w = z^2; \\ L = \{w: |w| = 1\} - double; \\ \alpha = 2\frac{a}{b} + 1 > 1 \end{pmatrix} =$$

$$= -\frac{16i}{b^2}\oint_L \frac{wdw}{(w^2 + 2\alpha w + 1)^2}.$$



We determine the singular points placed in the inner region of the closed contour L:

$$D_0 = \alpha^2 - 1 > 0;\ \ w_1 = -\alpha + \sqrt{D_0} > -1;$$

$$w_2 = -\alpha - \sqrt{D_0} < -1.$$

We use the integral Cauchy formula:

$$I = -\frac{16i}{b^2}\oint_L \frac{wdw}{(w-w_1)^2(w-w_2)^2} = \left(f(w) = \frac{w}{(w-w_2)^2}\right) =$$

$$= -\frac{16i}{b^2}\oint_L \frac{f(w)dw}{(w-w_1)^2} = -\frac{16i}{b^2}2\pi i f'(w_1) = \frac{32\pi}{b^2}f'(w_1);$$

$$f'(w) = -\frac{w+w_2}{(w-w_2)^3} \Rightarrow f'(w_1) = -\frac{w_1+w_2}{(w_1-w_2)^3} = \frac{2\alpha}{\left(2\sqrt{D_0}\right)^3}.$$

Finally we get

$$I = \frac{32\pi}{b^2}\frac{2\alpha}{\left(2\sqrt{D_0}\right)^3} = \frac{8\pi}{b^2}\frac{\alpha}{(\alpha^2-1)^{3/2}} = \frac{\pi(2a+b)}{[a(a+b)]^{3/2}}.$$

Problem 9 ***(10 points)***

Find the general solution of the system of equations

$$\begin{cases} \ddot{x} - 2\dot{x} - 2x - \ddot{y} + 4\dot{y} = 0, \\ 2\dot{x} + x - \ddot{y} - 4\dot{y} + 2y = 0. \end{cases}$$

Answer:

$$x = C_1 e^t + 5C_2 e^{-t} + 2C_3 e^{2t} + 2C_4 e^{-2t}, \qquad y = C_1 e^t + C_2 e^{-t} + C_3 e^{2t} + C_4 e^{-2t}.$$

Solution:

a)

Composing the sum of the equations of the system, we obtain the equation for the new function $z(t)$:

$$z = 2y - x, \qquad \ddot{z} - z = 0 \implies z = C_1 e^t + C_2 e^{-t} \implies x = 2y - C_1 e^t - C_2 e^{-t}.$$

Substituting $x(t)$ into one of the equations of the system, we obtain the equation for $y(t)$:

$$\ddot{y} - 4y = -3C_1 e^t + C_2 e^{-t}.$$

Find the general solution $\bar{y}$ of the corresponding homogeneous equation:

$$\bar{y} = C_3 e^{2t} + C_4 e^{-2t}.$$

Find a particular solution y^* of the non-homogeneous equation:

$$y^* = C_1 e^t - \frac{C_2}{3} e^{-t}.$$

Find a general solution y of the on-homogeneous equation:

$$y = \bar{y} + y^* = C_1 e^t - \frac{1}{3} C_2 e^{-t} + C_3 e^{2t} + C_4 e^{-2t}.$$

Substitute in the formula for x:

$$x = C_1 e^t - \frac{5}{3} C_2 e^{-t} + 2C_3 e^{2t} + 2C_4 e^{-2t}.$$

Re-assigning C_2, we finally get

$$x = C_1 e^t + 5C_2 e^{-t} + 2C_3 e^{2t} + 2C_4 e^{-2t}, \qquad y = C_1 e^t + C_2 e^{-t} + C_3 e^{2t} + C_4 e^{-2t}.$$

b)

We are looking for a solution in the form:

$$x = Ae^{kt}, \qquad y = Be^{kt}.$$

We substitute it into the system and after reducing by e^{kt} find non-zero solutions for A, B:

$$A(k^2 - 2k - 2) + B(-k^2 + 4k) = 0,$$
$$A(2k + 1) + B(-k^2 - 4k + 2) = 0.$$

Get the characteristic equation, find the roots:

$$(k^2 - 2k - 2)(-k^2 - 4k + 2) - (-k^2 + 4k)(2k + 1) = 0 \implies$$
$$\implies k^4 - 5k^2 + 4 = 0 \implies (k^2 - 1)(k^2 - 4) = 0 \implies k_{1,2} = \pm 1, \;\; k_{3,4} = \pm 2.$$

Then we find the corresponding similarity coefficients:

$$k_1 = 1 \Longrightarrow \ -3A + 3B = 0 \Longrightarrow \ A_1 = 1, \qquad B_1 = 1;$$
$$k_2 = -1 \Longrightarrow \ A - 5B = 0 \Longrightarrow \ A_2 = 5, \qquad B_2 = 1;$$
$$k_3 = 2 \Longrightarrow \ -2A + 4B = 0 \Longrightarrow \ A_3 = 2, \qquad B_3 = 1;$$
$$k_4 = -2 \Longrightarrow \ 6A - 12B = 0 \Longrightarrow \ A_4 = 2, \qquad B_4 = 1.$$

Finally, we compose the solution of the original system:

$$\binom{x}{y} = C_1 \binom{1}{1} e^{t} + C_2 \binom{5}{1} e^{-t} + C_3 \binom{2}{1} e^{2t} + C_4 \binom{2}{1} e^{-2t}.$$

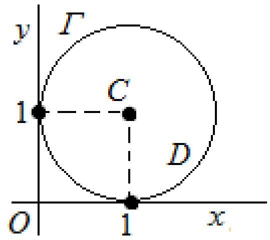
Problem 10 ***(9 points)***

Calculate the curvilinear integral

$$\oint_{\Gamma} \frac{xdy + ydx}{x^2 + y^2}, \quad \Gamma = \{(x, y) \colon (x-1)^2 + (y-1)^2 = 1\}.$$

Answer: 0.

Solution:



a)

Use Green 's formula

$$\int_{\Gamma} Pdx + Qdy = \iint_{D} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy, \qquad P = \frac{y}{x^2 + y^2}, \qquad Q = \frac{x}{x^2 + y^2}.$$

The area D is a circle and in polar coordinates ($x = r\cos\varphi$, $y = r\sin\varphi$) is defined by constraints:

$$D = \left\{ (r, \varphi) \colon 0 \le \varphi \le \frac{\pi}{2}, \ \ r_1(\varphi) \le r \le r_2(\varphi) \right\}$$

$$r_1(\varphi) = \cos\varphi + \sin\varphi - \sqrt{\sin 2\varphi}, \qquad r_2(\varphi) = \cos\varphi + \sin\varphi + \sqrt{\sin 2\varphi}$$

Then

$$\int_{\Gamma} \frac{xdy + ydx}{x^2 + y^2} = -2 \iint_{D} \frac{x^2 - y^2}{x^2 + y^2} dxdy = -2 \iint_{D} (\cos^2\varphi - \sin^2\varphi) r dr d\varphi =$$

$$= -2 \int_0^{\frac{\pi}{2}} \cos 2\varphi \, d\varphi \int_{r_1(\varphi)}^{r_2(\varphi)} r dr = -2 \int_0^{\frac{\pi}{2}} \cos 2\varphi \, d\varphi \left. \frac{r^2}{2} \right|_{r_1(\varphi)}^{r_2(\varphi)} =$$

$$= -4 \int_0^{\frac{\pi}{2}} \cos 2\varphi \, (\cos\varphi + \sin\varphi) \sqrt{\sin 2\varphi} \, d\varphi =$$

$$= -4\sqrt{2}\int_0^{\frac{\pi}{2}} \cos 2\varphi \cos\left(\varphi - \frac{\pi}{4}\right)\sqrt{\sin 2\varphi}\, d\varphi =$$

$$= \left(\theta = \varphi - \frac{\pi}{4}, -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}\right) = -4\sqrt{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin 2\theta \cos\theta \sqrt{\cos 2\theta}\, d\theta = 0,$$

due to the oddness of the integrand on a symmetric interval.

b)

Calculate immediately. Consider parametrization of the contour Γ:

$$x = 1 + \cos t, \qquad y = 1 + \sin t, \qquad 0 \leq t \leq 2\pi.$$

Then

$$dx = -\sin t\ dt, \qquad dy = \cos t\ dt$$

and we substitute everything into the integral:

$$\int_\Gamma \frac{xdy + ydx}{x^2 + y^2} = \int_0^{2\pi} \frac{(1+\cos t)\cos t - (1+\sin t)\sin t}{(1+\cos t)^2 + (1+\sin t)^2} dt =$$

$$= \int_0^{2\pi} \frac{(1+\cos t + \sin t)(\cos t - \sin t)}{3 + 2(\cos t + \sin t)} dt$$

$$= \int_0^{2\pi} \frac{1 + (\cos t + \sin t)}{3 + 2(\cos t + \sin t)} d(\cos t + \sin t) =$$

$$= \begin{pmatrix} u = \cos t + \sin t, \\ t = 0 \Longrightarrow u = 1, \\ t = 2\pi \Longrightarrow u = 1 \end{pmatrix} = \int_1^1 \frac{1+u}{3+2u} du = 0.$$
