INTERNATIONAL MATHEMATICAL OLYMPIAD
29.04.2021

Task 1 *(5 points)*

Calculate $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}^{2021}$.

Task 2 *(10 points)*

Find the least value of the function $u = 4x - 6y + 12z - 5$ in the set

$$\frac{x^2}{3^2} + \frac{y^2}{2^2} + \frac{z^2}{5^2} = 1.$$

Task 3 *(9 points)*

Find sum of the infinite series

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n!}.$$

Task 4 *(5 points)*

Find the limit of

$$\lim_{n\to\infty} \left(\cos\frac{x}{2} \cdot \cos\frac{x}{4} \cdot \ldots \cdot \cos\frac{x}{2^n}\right).$$

Task 5 *(5 points)*

Prove that the volume of a parallelepiped built on the faces diagonals of the parallelepiped is equal to twice the volume of this parallelepiped.

Task 6 *(6 points)*

Calculate the definite integral

$$\int_{1/a}^{a} \frac{lnx}{1+x^2}\,dx.$$

Task 7 *(5 баллов)*

The function is defined and satisfies the relation

$$(x-1)f\left(\frac{x+1}{x-1}\right) - f(x) = x$$

for all $x \in R,\ x \neq 1$. Find such functions.

Task 8 ***(9 points)***
Find a general solution to differential equation

$$y'' \cos x + y'(5 \cos x - 2 \sin x) + y(3 \cos x - 5 \sin x) = e^{-x}.$$

Task 9 ***(5 points)***
Prove that

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdot \ldots \cdot \frac{99}{100} < \frac{1}{10}.$$

Task 10 ***(11 points)***
Calculate the indefinite integral

$$I = \int \frac{x^2 dx}{(\sin x - x \cos x)^2}.$$

Task 11 ***(8 points)***
The numbers *p* and *q* are randomly selected on the segments $[2, 6], [0, 4]$ respectively. Find the probability that the roots of the equation $x^2 + px + q = 0$ are real and different.

Task 12 ***(9 points)***
Find solve of the Cauchy problem for differential equation

$$xyy'' - x(y')^2 = 2yy', \quad y(1) = e, \; y'(1) = 3e.$$

Task 13 ***(7 points)***
Prove that the polynomial

$$P(x) = x^n \sin \varphi - \rho^{n-1} x \sin n\varphi + \rho^n \sin(n-1)\varphi$$

is divisible by

$$x^2 - 2\rho x \cos \varphi + \rho^2.$$

Task 14 ***(6 points)***
Find a general solution to differential equation

$$y' + 2ye^x - y^2 = e^{2x} + e^x.$$