ЧЕТВЕРТАЯ МЕЖДУНАРОДНАЯ МАТЕМАТИЧЕСКАЯ ОЛИМПИАДА
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第四届国际数学奥林匹克竞赛
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Problem statements and solutions

Problem 1 ***(9 points)***

Prove the inequality

$$\left(1+\frac{1}{4}\right)\left(1+\frac{1}{8}\right)\ldots\left(1+\frac{1}{2^n}\right) < 2,\ \ n \geq 2.$$

Solution:

Let's logarithm the left side of the inequality and use the well-known fact: $\ln(1+x) < x, x \neq 0$, and the formula for the sum of geometric progression terms

$$\ln\left(\left(1+\frac{1}{4}\right)\left(1+\frac{1}{8}\right)\ldots\left(1+\frac{1}{2^n}\right)\right) = \sum_{k=2}^{n} \ln\left(1+\frac{1}{2^k}\right) < \sum_{k=2}^{n} \frac{1}{2^k} =$$

$$= \frac{1}{4}\cdot\frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}} < \frac{1}{2} < \ln 2.$$

Problem 2 ***(11 points)***

Calculate the indefinite integral

$$I = \int \frac{\cos x + \sin x}{5\cos^2 x - 2\sin 2x + 2\sin^2 x} dx.$$

Answer: $I = -\frac{3}{5}\operatorname{arctg}(2\cos x - \sin x) + \frac{1}{10\sqrt{6}} \ln\left|\frac{\sqrt{6}+(\cos x + 2\sin x)}{\sqrt{6}-(\cos x + 2\sin x)}\right| + C.$

Solution:
Заметим

$$5\cos^2 x - 2\sin 2x + 2\sin^2 x = (2\cos x - \sin x)^2 + 1 = 6 - (\cos x + 2\sin x)^2;$$

$$\sin x - \cos x = -\frac{3}{5}(2\cos x - \sin x) + \frac{1}{5}(\cos x + 2\sin x) \Longrightarrow$$

$$\Longrightarrow (\cos x + \sin x)dx = d(\sin x - \cos x)$$

$$= -\frac{3}{5}d(2\cos x - \sin x) + \frac{1}{5}d(\cos x + 2\sin x).$$

Transform the integral

$$I = \int \frac{\cos x + \sin x}{5\cos^2 x - 2\sin 2x + 2\sin^2 x} dx =$$

$$= -\frac{3}{5}\int \frac{d(2\cos x - \sin x)}{(2\cos x - \sin x)^2 + 1} + \frac{1}{5}\int \frac{d(\cos x + 2\sin x)}{6 - (\cos x + 2\sin x)^2} =$$

$$= -\frac{3}{5}\operatorname{arctg}(2\cos x - \sin x) + \frac{1}{10\sqrt{6}} \ln\left|\frac{\sqrt{6} + (\cos x + 2\sin x)}{\sqrt{6} - (\cos x + 2\sin x)}\right| + C.$$

Problem 3 ***(9 points)***
Calculate the definite integral

$$\int_0^1 \frac{x^{2023} - 1}{\ln x} dx.$$

Answer: $\ln 2024$.

Solution:
Consider a more general problem:

$$I(a,b)=\int_0^1 \frac{x^b-x^a}{\ln x}dx=\int_0^1 dx\int_a^b x^y\,dy=\int_a^b dy\int_0^1 x^y dx=\int_a^b \frac{x^y}{y+1}\Big|_0^1 dy$$
$$=\int_a^b \frac{dy}{y+1}=\ln\frac{b+1}{a+1}.$$

Then

$$\int_0^1 \frac{x^{2023}-1}{\ln x}dx=\ln 2024\,.$$

Problem 4 ***(10 points)***

Find a sum of the number series

$$\sum_{n=1}^{\infty}\frac{n^2}{3^n}.$$

Answer: 3/2.

Solution:

a)

Obviously, this series converges. We use the independence property of the sum of a convergent series (positive) from the permutation of its terms and the formula for the sum of an infinitely decreasing geometric progression

$$\sum_{n=1}^{\infty}\frac{n^2}{3^n}=\frac{1}{3}+\frac{4}{9}+\frac{9}{27}+\frac{16}{81}+\frac{25}{243}+\cdots=$$
$$=\frac{1}{3}+\left(\frac{1}{9}+\frac{3}{9}\right)+\left(\frac{1}{27}+\frac{3}{27}+\frac{5}{27}\right)+\left(\frac{1}{81}+\frac{3}{81}+\frac{5}{81}+\frac{7}{81}\right)+\cdots=$$
$$=\left(\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\cdots\right)+3\left(\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\cdots\right)+5\left(\frac{1}{27}+\frac{1}{81}+\cdots\right)+\cdots=$$
$$=\frac{1}{2}+3\cdot\frac{1}{6}+5\cdot\frac{1}{18}+7\cdot\frac{1}{54}+\cdots=\frac{1}{2}\cdot\left(1+\frac{3}{3}+\frac{5}{9}+\frac{7}{27}+\frac{9}{81}+\cdots\right)=$$
$$=\frac{1}{2}\cdot\left(\left(1+\frac{1}{3}+\frac{1}{9}+\cdots\right)+2\left(\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots\right)+2\left(\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\cdots\right)+\cdots\right)=$$
$$=\frac{1}{2}\cdot\left(\frac{3}{2}+2\cdot\frac{1}{2}+2\cdot\frac{1}{6}+\cdots\right)=\frac{1}{2}\cdot\left(\frac{3}{2}+1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots\right)=\frac{1}{2}\cdot\left(\frac{3}{2}+\frac{3}{2}\right)=\frac{3}{2}.$$

b)
Denote

$$S_0 = \sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{3}\frac{1}{1-\frac{1}{3}} = \frac{1}{2}$$

$$S_1 = \sum_{n=1}^{\infty} \frac{n}{3^n} = \sum_{n=0}^{\infty} \frac{n+1}{3^{n+1}} = \frac{1}{3}\sum_{n=1}^{\infty} \frac{n}{3^n} + \sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{3}S_1 + S_0 \Rightarrow$$

$$\Rightarrow \frac{2}{3}S_1 = S_0 \Rightarrow S_1 = \frac{3}{2}S_0 = \frac{3}{4}.$$

$$S_2 = \sum_{n=1}^{\infty} \frac{n^2}{3^n} = \sum_{n=0}^{\infty} \frac{(n+1)^2}{3^{n+1}} = \frac{1}{3}\sum_{n=1}^{\infty} \frac{n^2}{3^n} + \frac{2}{3}\sum_{n=1}^{\infty} \frac{n}{3^n} + \sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{3}S_2 + \frac{2}{3}S_1 + S_0 \Rightarrow$$

$$\Rightarrow \frac{2}{3}S_2 = \frac{2}{3}S_1 + S_0 \Rightarrow S_2 = S_1 + \frac{3}{2}S_0 = \frac{3}{2}.$$

Problem 5* *(9 points)

Find all real solutions to the equation

$$(2x^3 + x - 3)^3 = 3 - x^3.$$

Answer: $\sqrt[3]{3/2}$ – один действительный корень.

Solution:
Consider the function

$$f(x) = (2x^3 + x - 3)^3 + x^3 - 3;$$

its derivative function

$$f'(x) = 3(2x^3 + x - 3)^2(6x^2 + 1) + 3x^2 > 0,$$

so the function $f(x)$ increases monotonically along the entire numerical axis from $-\infty$ to $+\infty$ and has a single real root. Transform the left side of the equation

$$(2x^3 + x - 3)^3 = (2x^3 - 3)^3 + 3x(2x^3 - 3)^2 + 3x^2(2x^3 - 3) + x^3.$$

The original equation is reduced to the form

$$(2x^3 - 3)((2x^3 - 3)^2 + 3x(2x^3 - 3) + 3x^2 + 1) = 0,$$

from where the only root $x = \sqrt[3]{3/2}$ is determined.

Problem 6 ***(9 points)***

Find a conditional extremum of the function

$$u = xyz$$

with the following coupling equations

$$xy + yz + zx = 9,\ \ x + y + z = 6.$$

Answer: $u_{max} = u(1,4,1) = u(4,1,1) = u(1,1,4) = 4$; $u_{min} = u(0,3,3) = u(3,0,3) = u(3,3,0) = 0$.

Solution:

From the system of communication equations, we find

$$\begin{cases} xy + yz + zx = 9, \\ x + y + z = 6, \end{cases} \Rightarrow \begin{cases} xy = 9 - (x + y)z, \\ x + y = 6 - z. \end{cases}$$

We get $xy = 9 - (6 - z)z$, then the objective function takes the form

$$u = xyz = (9 - (6 - z)z)z = z^3 - 6z^2 + 9z = \varphi(z).$$

Investigate the function $\varphi(z)$ for an unconditional extremum. Let's find the first order derivative and stationary points:

$$\varphi'(z) = 3z^2 - 12z + 9 = 0 \qquad \Rightarrow \qquad z_1 = 1,\ \ z_2 = 3.$$

Obviously, the first stationary point is the maximum point for the function $\varphi(z)$, because

$$\varphi''(1) = (6z - 12)|_{z=1} = -6 < 0.$$

This point corresponds to two pairs x and y – $x_{1,1} = 1$, $y_{1,1} = 4$ and $x_{1,2} = 4$, $y_{1,2} = 1$. Thus, we obtain two points of conditional maximum – $M_1(1,4,1)$ and $M_2(4,1,1)$. Obviously, due to the symmetry of the connection conditions and the objective function, we will also have a third conditional maximum point $M_3(1,1,4)$ (this point can be obtained by expressing from the system of equations of coupling xz or yz). The conditional maximum value is equal to

$$u_{max} = u(1,4,1) = u(4,1,1) = u(1,1,4) = \varphi(1) = 4.$$

Similarly, the second stationary point is the minimum point for the function $\varphi(z)$, because

$$\varphi''(3) = (6z - 12)|_{z=3} = 6 > 0.$$

This point corresponds to two pairs x и y – $x_{2,1} = 0$, $y_{2,1} = 3$ and $x_{2,2} = 3$, $y_{2,2} = 0$. Thus, we obtain two points of conditional minimum – $N_1(0,3,3)$ and $N_2(3,0,3)$. And due to the symmetry of the connection conditions and the objective function, we will also have a third conditional minimum point $N_3(3,3,0)$. The conditional minimum value is equal to

$$u_{min} = u(0,3,3) = u(3,0,3) = u(3,3,0) = \varphi(3) = 0.$$

Problem 7 ***(10 points)***
Find a general solution to the differential equation

$$y^2dx + (e^x - y)dy = 0.$$

Answer: $e^x \ln(Cy) = y.$

Solution:
Make a replacement $e^x = z,\ dz = e^x dx = zdx$. Then

$$y^2 \frac{dz}{z} + (z - y)dy = 0 \Longrightarrow \frac{dz}{dy} = -\frac{z^2 - zy}{y^2}.$$

$$z = ty, \qquad \frac{dz}{dy} = \frac{dt}{dy}y + t \Longrightarrow \frac{dt}{dy}y + t = -t^2 + t \Longrightarrow \frac{1}{t} = \ln(Cy) \Longrightarrow e^x = \frac{y}{\ln(Cy)}.$$

Problem 8 ***(11 points)***
Find a general solution to the system of nonlinear differential equations

$$\begin{cases} y' = \dfrac{z}{x}, \\ z' = \dfrac{z(y + 2z - 1)}{x(y - 1)}. \end{cases}$$

Answer: $y(x) = \frac{C_2x - C_1 - 1}{C_2x - C_1}, \qquad z(x) = \frac{C_2x}{(C_2x - C_1)^2}.$

Solution:
Eliminate x and go to the function $z = z(y)$:

$$\frac{dz}{dy} = \frac{y + 2z - 1}{y - 1} \qquad \Longrightarrow \qquad \frac{dz}{dy} = \frac{2z}{y - 1} + 1.$$

We solve the resulting equation using the Bernoulli method:

$$z(y) = u(y)v(y),\ z' = u'v + uv' \Longrightarrow u'v + uv' = \frac{2uv}{y - 1} + 1.$$

We select the function $v(y)$ so that

$$v' = \frac{2v}{y - 1} \Longrightarrow \frac{dv}{v} = \frac{2dy}{y - 1} \Longrightarrow v = (y - 1)^2.$$

After this we have

$$u'(y-1)^2 = 1 \implies u' = \frac{1}{(y-1)^2} \implies u = -\frac{1}{y-1} + C_1 \implies$$
$$\implies z = uv = C_1(y-1)^2 - (y-1).$$

Next we return to the independent variable x:

$$y'x = C_1(y-1)^2 - (y-1) \implies \frac{dy}{C_1(y-1)^2 - (y-1)} = \frac{dx}{x} \implies$$
$$\implies \left(\frac{C_1}{C_1(y-1)-1} - \frac{1}{y-1}\right)dy = \frac{dx}{x} \implies \ln\frac{C_1(y-1)-1}{y-1} = \ln(C_2x) \implies$$
$$\implies \frac{C_1(y-1)-1}{y-1} = C_2x \implies y-1 = -\frac{1}{C_2x - C_1} \implies y = \frac{C_2x - C_1 - 1}{C_2x - C_1}.$$

All that remains is to find $z(x)$:

$$z(x) = y' \cdot x = \frac{C_2x}{(C_2x - C_1)^2}.$$

Problem 9 ***(11 points)***

Calculate n-th order determinant

$$\begin{vmatrix} 3 & 8 & 0 & 0 & \cdots & 0 & 0 \\ 2 & 6 & 9 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 6 & 9 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 6 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 6 & 9 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 6 \end{vmatrix}.$$

Answer: $3^{n-2}(16-7n)$.

Solution:

Denote Δ_k – k-th order principal diagonal minor. Then

$$\Delta_1 = 3,\ \Delta_2 = \begin{vmatrix} 3 & 8 \\ 2 & 6 \end{vmatrix} = 2,\ \Delta_3 = 6\Delta_2 - 9\Delta_1 = 3(2\Delta_2 - 3\Delta_1),$$
$$\Delta_4 = 6\Delta_3 - 9\Delta_2 = 3(2\Delta_3 - 3\Delta_2) = 3^2(3\Delta_2 - 6\Delta_1),$$
$$\Delta_5 = 6\Delta_4 - 9\Delta_3 = 3(2\Delta_4 - 3\Delta_3) = 3^3(4\Delta_2 - 9\Delta_1),$$

then by induction we find

$$\Delta_k = 6\Delta_{k-1} - 9\Delta_{k-2} = 3^{k-2}[(k-1)\Delta_2 - 3(k-2)\Delta_1].$$

Finally

$$\Delta_n = 3^{n-2}[(n-1)\Delta_2 - 3(n-2)\Delta_1] = 3^{n-2}(16-7n),$$

where n is the order of the determinant.

Problem 10 ***(11 points)***

Random variables ξ_1, ξ_2, ξ_3 are independent and uniformly distributed on the interval $[0;\ 1]$.

Find the probability $P(\xi_1 + \xi_2 + \xi_3 \leq x)$ for $x \geq 0$.

Answer: $\frac{x^3}{6}$ for $0 \leq x \leq 1$; $\frac{[x^3 - 3(x-1)^3]}{6}$ for $1 \leq x \leq 2$; $1 - \frac{(3-x)^3}{6}$ for $2 \leq x \leq 3$; 1 при $x \geq 3$.

Solution:

Represent the set of possible values of random variables ξ_1, ξ_2, ξ_3 in the form of a unit cube in R^3. Then the desired probability is defined as the ratio of the volume of a part of a unit cube, limited by an inclined plane $\xi_1 + \xi_2 + \xi_3 = x$, o the volume of a unit cube, that is, to unity. Namely:

$$P(\xi_1 + \xi_2 + \xi_3 \leq x) = \begin{cases} x^3/6 \text{ при } 0 \leq x \leq 1; \\ [x^3 - 3(x-1)^3]/6 \text{ при } 1 \leq x \leq 2; \\ 1 - (3-x)^3/6 \text{ при } 2 \leq x \leq 3; \\ 1 \text{ при } x \geq 3. \end{cases}$$